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(Quantum) twisted Yangians: symmetry, Baxterisation and centralizers

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Abstract

Based on the (quantum) twisted Yangians, integrable systems with special boundary conditions, called soliton non-preserving (SNP), may be constructed. In the present article we focus on the study of subalgebras of the (quantum) twisted Yangians, and we show that such a subalgebra provides an exact symmetry of the rational transfer matrix. We discuss how the spectrum of a generic transfer matrix may be obtained by focusing only on two types of special boundaries. It is also shown that the subalgebras, emerging from the asymptotics of tensor product representations of the (quantum) twisted Yangian, turn out to be dual to the (quantum) Brauer algebra. To deal with general boundaries in the trigonometric case we propose a new algebra, which also provides the appropriate framework for the Baxterisation procedure in the SNP case.

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1 Introduction

It is well established by now that, for any gl_n algebra or the corresponding quantum deformation ($\mathcal{U}_q(gl_n)$), one may consider integrable lattice models or quantum field theories with two distinct types of boundary conditions known as soliton preserving (SP) with underlying algebra the reflection algebra [1, 2] or soliton non-preserving (SNP) with underlying algebra the twisted Yangian [3] or quantum twisted Yangian [4]. Historically, soliton non-preserving boundary conditions were first introduced in the context of affine Toda field theories on the half line [5], whereas solutions of the (quantum) twisted Yangian first found in [6]. Nonetheless up to date, both SP and SNP boundary conditions have been extensively studied in the context of integrable quantum spin chains (see e.g. [7, 8, 9]).

In this paper, we focus on the SNP case and we provide a generic description of the underlying symmetry algebra. The rational and trigonometric cases are considered separately. For the rational case, we investigate the symmetry and we show that the algebra, emerging from the twisted Yangian as $|\lambda| \to \infty$, is isomorphic to $so(p, q)$ or $sp(n)$ and is an exact symmetry of the algebraic open transfer matrix for a particular relation between the left and right boundaries. In addition, for this particular relation between the boundaries, we show that it is sufficient to study the spectrum of the transfer matrix for only two special cases. Finally, we recall the duality between the Brauer algebra and this algebra [10].

Similar considerations are made for the trigonometric case. In this case, the corresponding finite twisted quantum algebra, obtained from the quantum twisted Yangian for $|\lambda| \to \infty$, does not seem to provide an exact symmetry of the corresponding open model. We recall however the link between the finite twisted quantum algebra and the quantum Brauer algebra established in [11] for a trivial boundary. In the case of a generic boundary, we need to introduce a new algebra. We prove that a subalebra of this new algebra is the centralizer of the finite quantum twisted Yangian. The new framework allows us to find the spectral depending solution via a Baxterisation procedure.

The outline of this paper is as follows: In the next section, we give the algebraic setting by introducing the FRT presentation [12] for quantum groups and some subalgebras. We show for the rational case that the set of non local charges emerging from the asymptotic expansion of the tensor solution of the twisted Yangian, turn out to provide an exact symmetry of the transfer matrix. We then prove that the spectrum of any transfer matrix with generic boundary conditions can be deduced from the transfer matrix with special diagonal boundaries. We show that the entailed symmetry algebra commutes with appropriate representations of the Brauer algebra, depending on the choice of boundary conditions. In the last section, the trigonometric case is considered. Surprisingly the emerging boundary quantum algebra is not an exact symmetry of the trigonometric transfer matrix contrary to the rational case. Nevertheless, for trivial boundary conditions, we recall the duality between the quantum Brauer algebra and the twisted boundary quantum algebra. We finally provide a general framework, which allows us to deal with generic boundary conditions, and also discuss about the Baxterisation procedure.

2 (Quantum) twisted Yangian: general setting

Let $R(\lambda)$ be a solution of the Yang-Baxter equation [13, 14, 15]

$$R_{12}(\lambda_1 - \lambda_2)\ R_{13}(\lambda_1)\ R_{23}(\lambda_2) = R_{23}(\lambda_2)\ R_{13}(\lambda_1)\ R_{12}(\lambda_1 - \lambda_2), \tag{2.1}$$

acting on $\mathrm{End}(\mathbb{C}^n)^{\otimes 3}$, and as usual $R_{12}(\lambda) = R(\lambda) \otimes \mathbb{I}$, $R_{23}(\lambda) = \mathbb{I} \otimes R(\lambda)$ and so on. In addition, the R matrices we shall deal with satisfy
(i) the unitarity condition

$$R_{12}(\lambda)\ R_{21}(-\lambda) \propto \mathbb{I} \tag{2.2}$$

(ii) the crossing relation

$$R_{12}^{t_1}(\lambda)\ M_1\ R_{12}^{t_2}(-\lambda - 2i\rho)\ M_1^{-1} \propto \mathbb{I}, \tag{2.3}$$

where t_i denotes the transposition on the i^{th} space
(iii) the symmetry

$$\Big[M_1\ M_2,\ R_{12}(\lambda)\Big] = 0. \tag{2.4}$$

Let us also define the matrix

$$\bar{R}_{12}(\lambda) \propto R_{12}^{t_1}(-\lambda - i\rho)\,, \tag{2.5}$$

which will be useful for our purposes here. The proportional factors have no influence on the definition of the algebra and will be chosen conveniently.

For an associative infinite algebra $\mathcal{Y}$ generated by $\{L_{ij}^{(m)} | 1 \leqslant i,j \leqslant n, m = 0,1,\dots\}$, one may define the object

$$\mathbb{L}(\lambda) = \sum_{i,j=1}^{n} E_{ij} \otimes L_{ij}(\lambda) \in \mathrm{End}(\mathbb{C}^n) \otimes \mathcal{Y}[\lambda^{-1}]\,, \tag{2.6}$$

where the second 'space' is occupied by formal series $L_{ij}(\lambda) = \displaystyle\sum_{m=0}^{+\infty} \frac{L_{ij}^{(m)}}{\lambda^m}$ and E_{ij} is a n by n matrix such that $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. The exchange relations may be written using the so-called FRT relation [12],

$$R_{12}(\lambda_1 - \lambda_2)\ \mathbb{L}_{13}(\lambda_1)\ \mathbb{L}_{23}(\lambda_2) = \mathbb{L}_{23}(\lambda_2)\ \mathbb{L}_{13}(\lambda_1)\ R_{12}(\lambda_1 - \lambda_2), \tag{2.7}$$

where the 1 and 2 stand for the two copies of $\mathrm{End}(\mathbb{C}^n)$ whereas 3, for the space $\mathcal{Y}[\lambda^{-1}]$.

In this article, we will be interested in two particular choices of the algebra called the $gl(n)$ (quantum) Yangian [16, 17] depending on the choice of a rational or trigonometric R matrix.

In this type of algebra defined by FRT relation, we can exhibit different homomorphisms. In particular, we will use $\mathbb{L}(\lambda) \mapsto \bar{\mathbb{L}}(\lambda) = f(\lambda)\mathbb{L}^t(-\lambda - i\rho)$ i.e. in terms of generators (entries of $\mathbb{L}$)

$$L_{ij}(\lambda) \mapsto \bar{L}_{ij}(\lambda) = f(\lambda) L_{ji}(-\lambda - i\rho) \tag{2.8}$$

where $f(\lambda)$ is any function chosen later conveniently.

The (quantum) Yangian defined by relation (2.7) are also Hopf algebras. In particular, they are equipped with a coproduct $\Delta : \mathcal{Y} \to \mathcal{Y} \otimes \mathcal{Y}$ given by, for $i,\ j \in \{1, \dots, n\}$,

$$\Delta(L_{ij}(\lambda)) = \sum_{a=1}^{n} L_{aj}(\lambda) \otimes L_{ia}(\lambda) \tag{2.9}$$

which can be written also as

$$(\mathrm{id} \otimes \Delta)\mathbb{L}(\lambda) = \mathbb{L}_{02}(\lambda)\ \mathbb{L}_{01}(\lambda)\,. \tag{2.10}$$

The index 0 stands for the space $\mathrm{End}(\mathbb{C}^n)$, whereas 1 and 2 stand for the two copies of the algebra. This coproduct induces also

$$(\mathrm{id} \otimes \Delta)\bar{\mathbb{L}}(\lambda) = \bar{\mathbb{L}}_{01}(\lambda)\ \bar{\mathbb{L}}_{02}(\lambda)\,. \tag{2.11}$$

The ℓ-coproduct $\Delta^{(\ell)} : \mathcal{Y} \to \mathcal{Y}^{\otimes \ell}$ is defined by the following iteration $\Delta^{(\ell)} = \left(\mathrm{id} \otimes \Delta^{(\ell-1)}\right)\Delta$ with $\Delta^{(2)} = \Delta$.

In order to deal with integrable systems with non trivial boundary conditions, one has to consider appropriate subalgebras of $\mathcal{Y}$. For this purpose, we define

$$\mathbb{K}_0(\lambda) = \sum_{i,j=1}^{n} E_{ij} \otimes K_{ij}(\lambda) = \mathbb{L}_{01}(\lambda)\ \mathcal{K}_0(\lambda)\bar{\mathbb{L}}_{01}(\lambda), \tag{2.12}$$

with $\mathcal{K}(\lambda)$ a c-number matrix solution of the following equation

$$R_{12}(\lambda_1 - \lambda_2)\ \mathcal{K}_1(\lambda_1)\ \bar{R}_{12}(\lambda_1 + \lambda_2)\ \mathcal{K}_2(\lambda_2) = \mathcal{K}_2(\lambda_2)\ \bar{R}_{12}(\lambda_1 + \lambda_2)\ \mathcal{K}_1(\lambda_1)\ R_{12}(\lambda_1 - \lambda_2) \tag{2.13}$$

The generators encompassed in $\mathbb{K}(\lambda)$ generate a subalgebra $\mathcal{T}$ of $\mathcal{Y}$, whose exchange relations are given by (2.13), where now $\mathcal{K}(\lambda)$ is replaced by $\mathbb{K}(\lambda)$. Depending on the choice of the R matrix, $\mathcal{T}$ is the (quantum) twisted Yangian[3] [3, 4]. It is clear that $\mathbb{K}(\lambda)$ allows the expansion in powers of λ^{-1} as we shall see in subsequent section, providing explicit forms of the generators of the twisted Yangian.

The subalgebra $\mathcal{T}$ is not a Hopf algebra but has a structure of co-ideal inherited essentially from the (quantum) Yangian, $\Delta : \mathcal{T} \to \mathcal{T} \otimes \mathcal{Y}$, such that (see also [18, 19]), for $i,\ j \in \{1, \dots, n\}$,

$$\Delta(K_{ij}(\lambda)) = \sum_{a,b=1}^{n} K_{ab}(\lambda) \otimes L_{ia}(\lambda)\bar{L}_{bj}(\lambda) \tag{2.14}$$

[3]In the case of the twisted Yangian, a supplementary relation is required for $\mathbb{K}$ (see (3.7))

or, equivalently, $(\mathrm{id}\otimes\Delta)\mathbb{K}(\lambda) = \mathbb{L}_{02}(\lambda)\mathbb{L}_{01}(\lambda)\,\mathcal{K}_0(\lambda)\bar{\mathbb{L}}_{01}(\lambda)\bar{\mathbb{L}}_{02}(\lambda)$. One can exploit the existence of tensor product realizations of the (quantum) twisted Yangian in order to build the corresponding quantum system that is the open quantum spin chain with SNP boundary conditions. We define

$$\mathbb{T}_0(\lambda) = (\mathrm{id}\otimes\Delta^{(N)})\mathbb{L}(\lambda) = \mathbb{L}_{0N}(\lambda)\ldots\mathbb{L}_{01}(\lambda)\,, \tag{2.15}$$

$$\bar{\mathbb{T}}_0(\lambda) = (\mathrm{id}\otimes\Delta^{(N)})\bar{\mathbb{L}}(\lambda) = \bar{\mathbb{L}}_{01}(\lambda)\ldots\bar{\mathbb{L}}_{0N}(\lambda)\,. \tag{2.16}$$

Then, the general tensor type solution of the (2.13) takes the form

$$\mathbb{B}_0(\lambda) = (\mathrm{id}\otimes\Delta^{(N)})\mathbb{K}(\lambda) = \mathbb{T}_0(\lambda)\,\mathcal{K}_0^{(R)}(\lambda)\,\bar{\mathbb{T}}_0(\lambda)\,, \tag{2.17}$$

where $\mathcal{K}_0^{(R)}(\lambda)$ is a c-number matrix solution of (2.13) interpreted as the reflection on the right boundary. The entries $B_{ij}(\lambda)$ of the matrix $\mathbb{B}(\lambda)$ can be computed explicitly by $B_{ij}(\lambda) = \Delta^{(N)}(K_{ij}(\lambda))$.

Finally, we introduce the transfer matrix of the open spin chain [2], which may be written as

$$t(\lambda) = \mathrm{tr}_0\ \Big\{\mathcal{K}_0^{(L)}(\lambda)\ \mathbb{B}_0(\lambda)\Big\}\,, \tag{2.18}$$

where $\mathcal{K}^{(L)}(\lambda) = \mathcal{K}(-\lambda - i\rho)$ encodes the interaction with the left boundary and $\mathcal{K}(\lambda)$ is a solution of (2.13). It can be shown [2, 7], using the fact that $\mathbb{B}(\lambda)$ is a solution of (2.13), that transfer matrix (2.18) provides family of commuting operators i.e.,

$$\Big[t(\lambda),\ t(\lambda')\Big] = 0\,. \tag{2.19}$$

The latter commutation relation (2.19) ensures the integrability of the relevant models.

3 The rational case

One of the main objectives of the present study is the derivation of the exact symmetry for integrable models associated to (quantum) twisted Yangian. We shall first examine the rational case and, in the subsequent section, we shall proceed with the trigonometric case.

The gl_n R matrix is written in the following simple form,

$$R(\lambda) = \mathbb{I} + \frac{i}{\lambda}\mathcal{P} \tag{3.1}$$

where $\mathcal{P} = \sum_{a,b=1}^{n} E_{ab}\otimes E_{ba}$ is the permutation operator, acting on $(\mathbb{C}^n)^{\otimes 2}$. Note that for the rational case $\rho = \frac{n}{2}$ and $M = \mathbb{I}$ (see (2.3)). Let us define $\check{\mathcal{P}} = \rho\mathbb{I} - \mathcal{Q}$ where $\mathcal{Q} = \sum_{a,b=1}^{n} E_{ab}\otimes E_{ab}$ is a one-dimensional projector satisfying

$$\mathcal{Q}\,\mathcal{P} = \mathcal{P}\,\mathcal{Q} = \mathcal{Q}, \qquad \mathcal{Q}^2 = n\mathcal{Q}, \tag{3.2}$$

and consequently $\check{\mathcal{P}}^2 = \rho^2 \mathbb{I}$. Then, the $\bar{R}$ matrix can be written as

$$\bar{R}(\lambda) = \mathbb{I} + \frac{i}{\lambda} \check{\mathcal{P}} \tag{3.3}$$

For the rational case, a solution of the fundamental equation (2.7) is provided by evaluation map i.e.

$$L_{ij}(\lambda) = \delta_{ij} + \frac{ie_{ji}}{\lambda}, \qquad \bar{L}_{ij}(\lambda) = \frac{\lambda + i\rho}{\lambda} L_{ji}(-\lambda - i\rho) = \delta_{ij} + \frac{i\rho\delta_{ij} - ie_{ij}}{\lambda} \tag{3.4}$$

where e_{ij} are the generators of the Lie algebra $gl(n)$ satisfying

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj} \tag{3.5}$$

Let us introduce $\mathbb{P} = \sum_{i,j} E_{ij} \otimes E_{ji}$ and $\check{\mathbb{P}} = \sum_{i,j} E_{ij} \otimes (\rho\delta_{ij} - E_{ij})$. Then, we can write

$$\mathbb{L}(\lambda) = 1 + \frac{i\mathbb{P}}{\lambda} \qquad \text{and} \quad \bar{\mathbb{L}}(\lambda) = 1 + \frac{i\check{\mathbb{P}}}{\lambda} \tag{3.6}$$

It is clear that $\mathcal{P}$ and $\check{\mathcal{P}}$ are the fundamental representations of the corresponding $\mathbb{P}$ and $\check{\mathbb{P}}$.

3.1 Symmetry of the transfer matrix

We shall also need in what follows the c–number solution of (2.13) for the rational case, which is any constant matrix (λ independent) such that $\mathcal{K} = \pm \mathcal{K}^t$ [8, 20]. We shall show that different choices of $\mathcal{K}$ provide different symmetry algebras. We suppose also that $\mathcal{K}$ is invertible and with real entries. We recall that there does not exist invertible antisymmetric $n \times n$ matrix for n odd (Jacobi's theorem easily checked by $\det(\mathcal{K}) = (-1)^n \det(\mathcal{K})$). For this choice of R matrix, the elements encompassed in $\mathbb{B}(\lambda)$ generated the twisted Yangian [3]. The elements satisfied the following supplementary symmetry relation[4]

$$\mathbb{B}^t(\lambda) = \pm \mathbb{B}(-\lambda - i\rho) g(\lambda) + \frac{i}{2\lambda + i\rho} \big(\mathbb{B}(-\lambda - i\rho) g(\lambda) - \mathbb{B}(\lambda) \big), \tag{3.7}$$

where the upper (resp. lower) sign corresponds to $\mathcal{K}$ symmetric (resp. antisymmetric) and $g(\lambda) = \left(\frac{\lambda + i\rho}{\lambda}\right)^{2N}$.

We shall consider here the expansion of $\mathbb{B}(\lambda)$ as $\lambda \to \infty$ in order to obtain explicit expressions of generators of the twisted Yangian (see also [21]). We shall keep up to $\frac{1}{\lambda}$ terms which is sufficient for our purposes. Bearing in mind the expansions of $\mathbb{L}(\lambda)$ and $\bar{\mathbb{L}}(\lambda)$, we may easily deduce the asymptotic behavior of $\mathbb{B}(\lambda)$ as $\lambda \to \infty$, i.e.

$$\mathbb{B}_0(\lambda \to \infty) = \mathcal{K}_0 + \frac{i}{\lambda} \mathbb{B}_0^{(1)} + \ldots = \mathcal{K}_0 + \frac{i}{\lambda} \sum_{i=1}^{N} (\mathcal{K}_0 \, \check{\mathbb{P}}_{0i} + \mathbb{P}_{0i} \, \mathcal{K}_0) + \ldots . \tag{3.8}$$

[4] The differences between the relation presented here and this one in [3] are due to the shift in the spectral parameter of the matrix $\bar{R}(\lambda)$ in the commutation relation (2.13) and to the factor in $\bar{\mathbb{L}}(\lambda)$.

Let us define the following combination emerging essentially from the asymptotic expansion of the generalized solution of (2.13)

$$\mathbb{K}^{(1)} = \mathcal{K}\ \check{\mathbb{P}} + \mathbb{P}\ \mathcal{K} \qquad \text{i.e} \quad K^{(1)}_{ab} = \rho \mathcal{K}_{ab} - \sum_{c=1}^{n} \left(\mathcal{K}_{ac} e_{cb} - e_{ca} \mathcal{K}_{cb}\right). \tag{3.9}$$

The non–local charges i.e. the entries of (3.8), may be written simply as coproducts of the twisted Yangian elements namely $\mathbb{B}^{(1)} = \Delta^{(N)}(\mathbb{K}^{(1)})$. In particular, the asymptotic expansion of (2.14) provides

$$\Delta(\mathbb{K}^{(1)}) \;=\; \mathbb{K}^{(1)} \otimes \mathbb{I} + \mathbb{I} \otimes \mathbb{K}^{(1)} \;=\; \mathcal{K}_0\ \check{\mathbb{P}}_{01} + \mathbb{P}_{01}\ \mathcal{K}_0 + \mathcal{K}_0 \check{\mathbb{P}}_{02} + \mathbb{P}_{02} \mathcal{K}_0\,. \tag{3.10}$$

The elements $K^{(1)}_{ab}$ are called primitive and, in particular, their coproduct is cocomutative.

In the following, we will study more precisely the algebra $\mathcal{T}^{(1)}$ spanned by $B^{(1)}_{ab}$ but, before that, we shall show that they commute with the transfer matrix of the system provided that special boundary conditions are considered. In the proof we shall simply exploit the underlying algebraic relations provided by (2.13) as $\lambda_1 \to \infty$ i.e.

$$\left[\ \mathbb{B}^{(1)}_1\ ,\ \mathbb{B}_2(\lambda)\ \right] = \mathcal{P}_{12}\big(\mathbb{B}_1(\lambda)\mathcal{K}_2 - \mathcal{K}_1 \mathbb{B}_2(\lambda)\big) + \mathcal{K}_1 \mathcal{Q}_{12} \mathbb{B}_2(\lambda) - \mathbb{B}_2(\lambda) \mathcal{Q}_{12} \mathcal{K}_1\,. \tag{3.11}$$

Then, we extract the element in the position (a, b) in the space 1 and (c, d) in the space 2

$$\left[\ B^{(1)}_{ab}\ ,\ B_{cd}(\lambda)\ \right] = \mathcal{K}_{ad}\ B_{cb}(\lambda) - \mathcal{K}_{cb}\ B_{ad}(\lambda) + \mathcal{K}_{ac}\ B_{bd}(\lambda) - \mathcal{K}_{db}\ B_{ca}(\lambda)\ . \tag{3.12}$$

From the latter exchange relations it is entailed that for a generic $\mathcal{K}^{(R)}$ matrix the elements $B^{(1)}_{ab}$ do not commute with the transfer matrix. If however we consider the case where the left boundary is $\mathcal{K}^{(L)} = \mathcal{K}^{(R)^{-1}}$, the transfer matrix becomes

$$t(\lambda) = \sum_{l,\ c=1}^{n} \mathcal{K}^{(L)}_{lc}\ B_{cl}(\lambda)\,. \tag{3.13}$$

From the latter equation and bearing in mind the exchange relations (3.12), it is easy to show that

$$\Big[t(\lambda),\ B^{(1)}_{ab}\Big] = 0. \tag{3.14}$$

Note that an alternative proof may be formulated along the lines of [7, 22]. The particular choice of the left boundary indicates that the two boundaries of the chain have to be appropriately tuned so that all the elements $B^{(1)}_{ab}$ commute with the open transfer matrix. In fact, the special cases: (i) $\mathcal{K}^{(L)}_{ab} = \delta_{ab} = \mathcal{K}^{(R)}_{ab}$, (ii) $\mathcal{K}^{(L)}_{ab} = \delta_{a,n+1-b} = \mathcal{K}^{(R)}_{ab}$, (iii) $\mathcal{K}^{(L)}_{ab} = (-1)^a\ i\ \delta_{a,n+1-b} = \mathcal{K}^{(R)}_{ab}$ fall to the category above. The symmetry of cases (ii) and (iii) has been studied also in [9]. One may extract valuable information concerning the exact symmetry of a chain with generic boundary conditions exploiting the algebraic relations (3.12).

Now, we describe more precisely the algebra $\mathcal{T}^{(1)}$ spanned by $\{B^{(1)}_{ab} | 1 \le a, b \le n\}$. We are going to show that in the case $\mathcal{K}$ symmetric it is isomorphic to the Lie algebra $so(p, q)$ whereas for $\mathcal{K}$ antisymmetric it is isomorphic to $sp(n)$. For convenience, we introduce the sign $\epsilon = \pm$ such that $\mathcal{K}^t = \epsilon\mathcal{K}$. By expanding expressions (3.7) and (3.12) up to λ^{-1}, we get

$$B^{(1)}_{ba} = -\epsilon B^{(1)}_{ab} + \epsilon 2\rho N \mathcal{K}_{ab} \tag{3.15}$$

$$\left[\, B^{(1)}_{ab} \,,\, B^{(1)}_{cd} \,\right] = \mathcal{K}_{ad}\, B^{(1)}_{cb} - \mathcal{K}_{cb}\, B^{(1)}_{ad} + \mathcal{K}_{ac}\, B^{(1)}_{bd} - \mathcal{K}_{db}\, B^{(1)}_{ca} \,. \tag{3.16}$$

Let us now introduce the following $n \times n$ matrices

$$\mathcal{G}^{+} = \mathrm{diag}(\underbrace{1, \ldots, 1}_{p}, \underbrace{-1, \ldots, -1}_{q}) \qquad \text{and} \quad \mathcal{G}^{-} = \mathrm{diag}(\underbrace{1, \ldots, 1}_{n/2}) \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \,, \tag{3.17}$$

where $p + q = n$ and the second case is valid only for n even. A well-known linear algebra result is that the matrices $\mathcal{G}^{\epsilon}$ are the normal forms of the matrix $\mathcal{K}$ over reals under congruence i.e there exists an invertible real matrix $\mathcal{U}$ such that

$$\mathcal{U}\mathcal{K}\mathcal{U}^t = \mathcal{G}^{\epsilon} \,. \tag{3.18}$$

The mapping

$$\mathbb{B}^{(1)} \quad \longmapsto \quad \epsilon\mathcal{U}(\mathbb{B}^{(1)} - \rho N \mathcal{K})\mathcal{U}^t\mathcal{G}^{\epsilon} = \mathcal{U}\mathbb{B}^{(1)}\mathcal{K}^{-1}\mathcal{U}^{-1} - \rho N = \mathbb{M} \tag{3.19}$$

$$\text{i.e.} \quad B^{(1)}_{ab} \quad \longmapsto \quad \epsilon \sum_{\alpha,\beta,\gamma=1}^{n} \mathcal{U}_{a\alpha}\big(B^{(1)}_{\alpha\beta} - \rho N \mathcal{K}_{\alpha\beta}\big)\mathcal{U}_{\gamma\beta}\mathcal{G}^{\epsilon}_{\gamma b} = M_{ab} \tag{3.20}$$

defines an algebra isomorphism from $\mathcal{T}^{(1)}$ to $so(p, q)$ (resp. to $sp(n)$) for $\epsilon = +$ (resp. $\epsilon = -$). The bijection is proven by the fact that $\mathcal{U}$ is invertible. By direct computation of the relations satisfying by M_{ab} starting from relations (3.15) and (3.16), we recognize the following defining relations of $so(p, q)$ (resp. $sp(n)$)

$$\mathbb{M}^t\, \mathcal{G}^{\epsilon} \;=\; -\mathcal{G}^{\epsilon}\, \mathbb{M} \tag{3.21}$$

$$[\mathbb{M}_1, \mathbb{M}_2] \;=\; [\mathbb{M}_2, \mathcal{P}_{12} \;-\; \mathcal{G}^{\epsilon}_1\, \mathcal{Q}_{12}\, (\mathcal{G}^{\epsilon}_1)^{-1}] \tag{3.22}$$

which show the algebra homomorphism.

3.2 Treatment of general boundary condition

Henceforth we focus on the fundamental representation i.e. $\mathbb{L}_{0i}(\lambda) \mapsto R_{0i}(\lambda)$ and $\bar{\mathbb{L}}_{0i}(\lambda) \mapsto \bar{R}_{0i}(\lambda)$ and we restrict ourselves to the case where $\mathcal{K}^{(R)} = {\mathcal{K}^{(L)}}^{-1} = \mathcal{K}$. Then, we get the following representation for $\mathbb{B}(\lambda)$,

$$\mathbb{B}_0(\lambda) \mapsto R_{0N}(\lambda) \ldots R_{01}(\lambda)\, \mathcal{K}_0\, \bar{R}_{01}(\lambda) \ldots \bar{R}_{0N}(\lambda) = \mathcal{B}_0(\lambda) \,. \tag{3.23}$$

In this case, we have the following important proposition which allows us to find the spectrum of the transfer matrix for any boundary condition by studying only the case where the boundary conditions are given by $\mathcal{G}^{\pm}$.

Proposition 1 *Let $\mathcal{K}$ be any solution of (2.13) and $t_{\mathcal{K}}(\lambda) = \mathrm{tr}_0(\mathcal{K}_0^{-1}\mathcal{B}_0(\lambda))$. There exists invertible matrix $\mathcal{U}$ such that $\mathcal{U}\mathcal{K}\mathcal{U}^t = \mathcal{G}^\epsilon$ where $\mathcal{G}^\epsilon$ are defined in (3.17). Let*

$$t_{\mathcal{G}^\epsilon}(\lambda) = \mathrm{tr}_0\left((\mathcal{G}_0^\epsilon)^{-1} R_{0N}(\lambda)\dots R_{01}(\lambda)\,\mathcal{G}_0^\epsilon\,\bar{R}_{01}(\lambda)\dots\bar{R}_{0N}(\lambda)\right) .$$

Then $t_{\mathcal{K}}(\lambda)$ and $t_{\mathcal{G}^\epsilon}(\lambda)$ have the same eigenvalues, their eigenvectors (say $V_{\mathcal{K}}$ and $V_{\mathcal{G}^\epsilon}$ respectively) being related trough

$$V_{\mathcal{G}^\epsilon} = \mathcal{U}_1\dots\mathcal{U}_N V_{\mathcal{K}} \ . \tag{3.24}$$

Proof. The proof of this proposition is based on the relation

$$\mathcal{U}_1\dots\mathcal{U}_N\ t_{\mathcal{K}}(\lambda) = t_{\mathcal{G}^\epsilon}(\lambda)\,\mathcal{U}_1\dots\mathcal{U}_N \tag{3.25}$$

which is obtained using the property $[\bar{R}_{0i}(\lambda), (\mathcal{U}_0^t)^{-1}\mathcal{U}_i] = 0$ and the symmetry relation (2.4). ∎

This result is similar to the one obtained in the soliton preserving case in [8, 23, 24, 25]. This proposition allows us to restrict the study of the spectrum of the transfer matrix only to the cases where the boundaries are given by $\mathcal{G}^\epsilon$. Let us emphasize that although the eigenvalues are identical the models obtained from $t_{\mathcal{K}}$ and $t_{\mathcal{G}^\epsilon}$ may be different.

3.3 Link with the Brauer algebra

We shall now focus on the relation between the symmetry algebra and the Brauer algebra [10]. A well-known presentation of the classical Brauer algebra $\mathcal{B}_N(\delta)$ is realized by $2N-2$ generators σ_i and τ_i $(1 \leq i \leq N-1)$ obeying exchange relations:

$$\begin{aligned}
\sigma_i^2 &= 1, \qquad \tau_i^2 = \delta\ \tau_i, \qquad \sigma_i\ \tau_i = \tau_i\ \sigma_i = \tau_i, \qquad i = 1,\dots,N-1\\
\sigma_i\ \sigma_j &= \sigma_j\ \sigma_i, \qquad \tau_i\ \tau_j = \tau_j\ \tau_i, \qquad \sigma_i\ \tau_j = \tau_j\ \sigma_i, \qquad |i-j| > 1\\
\sigma_i\ \sigma_{i+1}\ \sigma_i &= \sigma_{i+1}\ \sigma_i\ \sigma_{i+1}, \qquad \tau_i\ \tau_{i\pm1}\ \tau_i = \tau_i,\\
\sigma_i\tau_{i+1}\ \tau_i &= \sigma_{i+1}\ \tau_i, \qquad \tau_{i+1}\ \tau_i\ \sigma_{i+1} = \tau_{i+1}\ \sigma_i, \qquad i = 1,\dots N-2
\end{aligned} \tag{3.26}$$

By direct computation, it is clear that the following map

$$\sigma_i \mapsto \epsilon\ \mathcal{P}_{i\ i+1} \qquad \text{and} \qquad \tau_i \mapsto \mathcal{K}_i\ \mathcal{Q}_{i\ i+1}\ \mathcal{K}_i^{-1}\ , \tag{3.27}$$

is a representation of the Brauer algebra $\mathcal{B}_N(n)$ for any matrix $\mathcal{K}$ satisfying $\mathcal{K}^t = \epsilon\mathcal{K}$ with $\epsilon = \pm$.

By representing relation (3.8), the conserved quantities in the fundamental representation are given by

$$\mathcal{B}_0^{(1)} = \sum_{j=1}^{N}\Big(\mathcal{P}_{0j}\ \mathcal{K}_0 + \mathcal{K}_0\ \check{\mathcal{P}}_{0j}\Big) \tag{3.28}$$

and generate an algebra isomorphic to $so(p,q)$ and $sp(n)$ depending on the choice of the $\mathcal{K}$ matrix. Using relation (3.27), it is now straightforward to show that the conserved quantities are the centralizers of the Brauer algebra i.e.

$$[\epsilon\ \mathcal{P}_{i\ i+1}\ ,\ \mathcal{B}_0^{(1)}] = 0 \qquad \text{and} \quad [\mathcal{K}_i\ \mathcal{Q}_{i\ i+1}\ \mathcal{K}_i^{-1}\ ,\ \mathcal{B}_0^{(1)}] = 0\,. \tag{3.29}$$

A few comments are in order at this point. In the SP case, the presence of a non trivial right boundary modifies naturally the form of the non-local charges, nevertheless they still commute with the transfer matrix as long as the left boundary is trivial. Furthermore, these charges continue to be the centralizers of an extended Hecke algebra called the B-type Hecke algebra (see [26]). In the case we examine here, the boundary non-local charges consist a symmetry algebra for the transfer matrix as long as the left and right boundaries are closely interrelated and are also centralizers of the Brauer algebra. In this spirit, it seems pointless to consider a 'boundary' extension of the Brauer algebra analogously to the Hecke case. However it is possible to conceive a generalization of the classical Brauer algebra regarding the $\mathcal{K}$ matrix as a representation of the extra element of the 'extended' algebra. Indeed, let us consider a possible extension of the Brauer algebra by introducing two supplementary generators b and b^{-1}, satisfying

$$\mathrm{b}\ \mathrm{b}^{-1} = \mathrm{b}^{-1}\ \mathrm{b} = 1\ , \qquad \sigma_j\ b = b\ \sigma_j, \qquad \tau_j\ b = b\ \tau_j \qquad \text{for} \quad j \geq 2 \tag{3.30}$$

$$\sigma_1\ \mathrm{b}\ \sigma_1\ \mathrm{b} = \mathrm{b}\ \sigma_1\ \mathrm{b}\ \sigma_1, \tag{3.31}$$

$$\sigma_1\ \mathrm{b}\ \tau_1\ \mathrm{b} = \mathrm{b}\ \tau_1\ \mathrm{b}\ \sigma_1, \qquad \sigma_1\ \mathrm{b}^{-1}\ \tau_1\ \mathrm{b}^{-1} = \mathrm{b}^{-1}\ \tau_1\ \mathrm{b}^{-1}\ \sigma_1\ . \tag{3.32}$$

It is easy to prove that the following map

$$\sigma_i \mapsto \epsilon\mathcal{P}_{i\ i+1} \qquad , \quad \tau_i \mapsto \mathcal{Q}_{i\ i+1} \qquad , \quad \mathrm{b} \mapsto \mathcal{K}_1 \qquad \text{and} \quad \mathrm{b}^{-1} \mapsto \mathcal{K}_1^{-1} \tag{3.33}$$

is a representation of this extended Brauer algebra. In the SP case, the representation of the extended Hecke algebra is important since it provides by Baxterisation new solutions of the reflection algebra. In our case, unfortunately, the Baxterisation does not apply given that no numerical solution depending on the spectral paremeter of (2.13) exists.

The picture presented above offers a rather tempting interpretation of the $\mathcal{K}$ matrix in terms of the 'boundary' generator. However, another interpretation exists where the $\mathcal{K}$ matrix allows us to define new interaction in the bulk. Let us define, for any $\mathcal{K}^t = \epsilon\mathcal{K}$,

$$\bar{R}'_{12}(\lambda) \;\; = \;\; \mathcal{K}_1\bar{R}_{12}(\lambda)\mathcal{K}_1^{-1} = \mathcal{K}_2\bar{R}_{12}(\lambda)\mathcal{K}_2^{-1}\,. \tag{3.34}$$

The transfer matrix, in the case where $(\mathcal{K}^{(L)})^{-1} = \mathcal{K}^{(R)} = \mathcal{K}$, may be written as

$$t_{\mathcal{K}}(\lambda) = \mathrm{tr}_0\{R_{0N}(\lambda)\dots R_{01}(\lambda)\bar{R}'_{01}(\lambda)\dots\bar{R}'_{0N}(\lambda)\}\,. \tag{3.35}$$

The commutativity of the transfer matrix (which provides, as usual, the integrability of the system) is ensured by

$$R_{12}(\lambda_1-\lambda_2)\ \mathcal{B}'_1(\lambda_1)\ \bar{R}'_{12}(\lambda_1+\lambda_2)\ \mathcal{B}'_2(\lambda_2) = \mathcal{B}'_2(\lambda_2)\ \bar{R}'_{12}(\lambda_1+\lambda_2)\ \mathcal{B}'_1(\lambda_1)\ R_{12}(\lambda_1-\lambda_2) \tag{3.36}$$

where $\mathcal{B}'_0(\lambda) = \mathcal{B}_0(\lambda)\,\mathcal{K}_0^{-1}$. The crucial point here is that the first non trivial terms in the expansion of $\mathcal{B}'_0(\lambda)$ for $\lambda \to +\infty$ i.e.

$$\mathcal{B}'^{(1)}_0 = \sum_{j=1}^{N} \Big(\mathcal{P}_{0j} + \mathcal{K}_0\ \check{\mathcal{P}}_{0j}\mathcal{K}_0^{-1}\Big) = N\rho + \sum_{j=1}^{N} \Big(\mathcal{P}_{0j} - \mathcal{K}_0\ \mathcal{Q}_{0j}\ \mathcal{K}_0^{-1}\Big) \tag{3.37}$$

still provides the symmetry of the transfer matrix. In addition, it is still the centralizer of the representation of the Brauer algebra, i.e.

$$[\epsilon\ \mathcal{P}_{i\ i+1}\ ,\ \mathcal{B}'^{(1)}_0] = 0 \qquad \text{and} \quad [\mathcal{K}_i\ \mathcal{Q}_{i\ i+1}\ \mathcal{K}_i^{-1}\ ,\ \mathcal{B}'^{(1)}_0] = 0 \qquad i \in \{1, \dots, N-1\}. \tag{3.38}$$

This construction is more symmetrical in the sense that the same operator appears in the symmetry algebra (3.37) and in the representation of the Brauer algebra (3.27).

Within this frame of mind, it seems rather unnecessary to discuss about a 'boundary extension' of the Brauer algebra. To conclude we showed the duality between the symmetry algebra of the transfer matrix and the Brauer algebra for suitable representations, depending on the choice of boundaries.

4 The trigonometric case

The $\mathcal{U}_q(\widehat{gl}_n)$ R matrix derived in [27] may be written in a compact form as

$$R(\lambda) = 2\sinh(\lambda + i\mu)\ \mathcal{P} + 2\sinh(\lambda)\ \mathcal{P}\ U \quad \text{and} \quad U = \sum_{\substack{i,j=1\\ i\neq j}}^{n} (E_{ij} \otimes E_{ji} - q^{-sgn(i-j)} E_{ii} \otimes E_{jj}) \tag{4.1}$$

where $q = e^{i\mu}$. In this case, $\rho = \mu n/2$ and the matrix M (2.3) is defined as

$$M_{ij} = q^{n-2j+1}\ \delta_{ij}\ . \tag{4.2}$$

Defining $R_{12}[q] = R_{12} = \mathcal{P}(q + U)$, we can write the R matrix more symmetrically as follows

$$R(\lambda) = e^{\lambda} R_{12} - e^{-\lambda} R_{21}^{-1} = e^{\lambda} R_{12}[q] - e^{-\lambda} R_{12}[q^{-1}]^{t_1 t_2} \tag{4.3}$$

The $\bar{R}$ matrix is given by

$$\bar{R}(\lambda) = e^{-\lambda - i\rho} R_{12}^{t_1} - e^{\lambda + i\rho} (R_{21}^{-1})^{t_1} \tag{4.4}$$

In particular, we get $\bar{R}(-i\rho) = (q - q^{-1})\mathcal{Q}$. It is well-known that rational R matrix is a limit of trigonometric R matrix. Indeed, rescaling the spectral parameter $\lambda \to \mu\lambda$ in definition (4.1) of the trigonometric R matrix and taking the limit $\mu \to 0$, we get rational R matrix (3.1). This limit, called the scaling limit, will be useful to compare the results of sections 3 and 4.

A solution of equation (2.7), where now R is a trigonometric matrix given above, may be written in the following simple form (see also [28])

$$\mathbb{L}(\lambda) = e^{\lambda}\mathbb{L}^+ - e^{-\lambda}\mathbb{L}^-, \qquad \mathbb{L}^+ = \sum_{\substack{i,j=1\\ i\le j}}^n E_{ij}\otimes \ell^+_{ij}, \qquad \mathbb{L}^- = \sum_{\substack{i,j=1\\ i\ge j}}^n E_{ij}\otimes \ell^-_{ij}. \tag{4.5}$$

with the matrices $\mathbb{L}^+$ ($\mathbb{L}^-$) apparently upper (lower) triangular and ℓ^+_{ij}, ℓ^-_{ij} the generators of the finite quantum group $U_q(gl_n)$. Its exchange relations can be also written using the FRT presentation as follows

$$R_{12}\mathbb{L}^{\pm}_1\mathbb{L}^{\pm}_2 = \mathbb{L}^{\pm}_2\mathbb{L}^{\pm}_1 R_{12} \tag{4.6}$$

$$R_{12}\mathbb{L}^{+}_1\mathbb{L}^{-}_2 = \mathbb{L}^{-}_2\mathbb{L}^{+}_1 R_{12}. \tag{4.7}$$

We have the following additional constraints on diagonal elements, for $1\le i\le n$,

$$\ell^+_{ii}\ell^-_{ii} = \ell^-_{ii}\ell^+_{ii} = 1\,. \tag{4.8}$$

Similarly to the rational case, we introduce $\bar{\mathbb{L}}(\lambda) = \mathbb{L}(-\lambda - i\rho)^t$ which can be written as

$$\bar{\mathbb{L}}(\lambda) = e^{\lambda}\bar{\mathbb{L}}^+ - e^{-\lambda}\bar{\mathbb{L}}^-, \qquad \bar{\mathbb{L}}^+ = \sum_{\substack{i,j=1\\ i\le j}}^n E_{ij}\otimes \bar{\ell}^+_{ij}, \qquad \mathbb{L}^- = \sum_{\substack{i,j=1\\ i\ge j}}^n E_{ij}\otimes \bar{\ell}^-_{ij}\,, \tag{4.9}$$

where $\bar{\ell}^{\pm}_{ij} = -e^{\pm i\rho}\ \ell^{\mp}_{ji}$.

In [29], a classification of numerical solutions of equation (2.13) for trigonometric R matrix has been presented. More precisely, the general invertible solutions $\mathcal{K}(\lambda)$, up to a global factor, can be written as

$$\mathcal{K}(\lambda) = \mathcal{D}\mathcal{G}(\lambda)\mathcal{D} \tag{4.10}$$

where $\mathcal{D}$ is any invertible constant matrix and $\mathcal{G}(\lambda)$ is one of the following matrices

(i) $\mathcal{G}(\lambda) = \mathbb{I}$

(ii) $$\frac{\mathcal{G}(\lambda)}{\left(q^{\frac12}+q^{-\frac12}\right)} = e^{\lambda}q^{\frac{n}{4}}\left(\frac{q^{-1}}{q^{-1}+1}\sum_{i=1}^n \epsilon_i^2 E_{ii} + \sum_{\substack{i,j=1\\ i<j}}^n \epsilon_i\epsilon_j E_{ij}\right) \pm e^{-\lambda}q^{-\frac{n}{4}}\left(\frac{q}{q+1}\sum_{i=1}^n \epsilon_i^2 E_{ii} + \sum_{\substack{i,j=1\\ i>j}}^n \epsilon_i\epsilon_j E_{ij}\right)$$

where $\epsilon_i = \sqrt{(-1)^{in}}$ and by convention $\sqrt{1}=1$, $\sqrt{-1}=i$. The solution given here is linked to the solution given in [29] by transformation (4.10) where $\mathcal{D} = \sum_{i=1}^n \sqrt{(-1)^{in}}E_{ii}$ and by a global factor.

(iii) $$\mathcal{G}(\lambda) = e^{2\lambda}q^{\frac{n-1}{2}}E_{1n} - e^{-2\lambda}q^{-\frac{n-1}{2}}E_{n1} + \sum_{i=1}^{\frac{n-2}{2}}(q^{\frac12}E_{2i\ 2i+1} - q^{-\frac12}E_{2i+1\ 2i}), \qquad n \quad \text{even}.$$

(iv) $\mathcal{G}(\lambda) = \displaystyle\sum_{i=1}^{\frac{n}{2}} (q^{\frac{1}{2}} E_{2i-1\ 2i} - q^{-\frac{1}{2}} E_{2i\ 2i-1}) = \mathcal{G}_q^-, \qquad n \quad \text{even.}$

Notice that the solutions (i), (ii) coincide with previously known results found in [6, 18]. By the scaling limit, solutions (i) and (iv) becomes respectively $\mathcal{G}^+$ (for $q = 0$) and $\mathcal{G}^-$ (see relation (3.17)). Solution (iii) becomes $E_{1n} - E_{n1} + \sum_{i=1}^{\frac{n-2}{2}} E_{2i,2i+1} - E_{2i+1,2i}$ which is antisymmetric. Solution (ii), with the upper sign, becomes $\sum_{i,j=1}^{n} E_{ij}$ which is never invertible and, with the lower sign, $\sum_{i<j}(E_{ij} - E_{ji})$ which is antisymmetric. We recover different invertible solutions of the reflection equation with the rational R matrix and some non invertible solutions.

Finally, let us remark that each solution can be written as

$$\mathcal{G}(\lambda) = \mathcal{G}^+(\lambda, q) + \sigma\, \mathcal{G}^+(-\lambda, q^{-1})^t \tag{4.11}$$

where $\mathcal{G}^+(\lambda, q)$ is equal to (i) $\mathbb{I}/2$ ($\sigma = +$), (ii) $e^\lambda q^{\frac{n}{4}} \left(\frac{q}{q+1} \sum_{i=1}^{n} \epsilon_i^2 E_{ii} + \sum_{i<j} \epsilon_i \epsilon_j E_{ij} \right)$ ($\sigma = \pm$), (iii) $e^{2\lambda} q^{\frac{n-1}{2}} E_{1n} + \sum_{i=1}^{\frac{n-2}{2}} q^{\frac{1}{2}} E_{2i\ 2i+1}$ ($\sigma = -$) and (iv) $\sum_{i=1}^{\frac{n}{2}} q^{\frac{1}{2}} E_{2i-1\ 2i}$ ($\sigma = -$). Note the similarity between relations (4.3) and (4.11) and the fact that $\mathcal{G}^+(\lambda, q)$ is upper diagonal as R_{12}.

4.1 Finite subalgebras

The asymptotic expansion of $\mathbb{B}(\lambda)$ as $\lambda \to \pm\infty$ will provide explicit expressions of the finite quantum twisted Yangian generators. We shall keep λ independent expressions, bearing in mind the form of $\mathbb{L}$ (4.5) and $\bar{\mathbb{L}}$ (4.9) and also the generic solutions given above. It is clear that the first three solutions preserve the triangular decomposition of the $\mathbb{B}(\lambda \to \pm\infty) = \mathbb{B}^\pm = \sum_{i,j=1}^{n} E_{ij} \otimes B_{ij}^\pm$ matrix, while for the last one $\mathbb{B}^\pm$ is not triangular any more. Explicit expressions of $\mathbb{B}^+$ for the various solutions (i)–(iv) are given below

(i) For $i < j$, $B_{ij}^+ = \displaystyle\sum_{k=i}^{j} \ell_{ik}^+\ \bar{\ell}_{kj}^+$, $B_{ii}^+ = -e^{i\rho}$ and for $i > j$, $B_{ij}^+ = 0$.

(ii) For $i < j$, $B_{ij}^+ \propto \dfrac{q^{-1}}{q^{-1}+1} \displaystyle\sum_{k=i}^{j} \epsilon_k^2 \ell_{ik}^+\ \bar{\ell}_{kj}^+ + \sum_{k=i}^{j-1} \sum_{p=k+1}^{j} \epsilon_k \epsilon_p \ell_{ik}^+\ \bar{\ell}_{pj}^+$, $B_{ii}^+ \propto -\dfrac{\epsilon_i^2 e^{i\rho}\, q^{-1}}{q^{-1}+1}$ and for $i > j$, $B_{ij}^+ = 0$

(iii) $B_{1n}^+ \propto \ell_{11}^+ \bar{\ell}_{nn}^+$ and 0 otherwise.

(iv) $B_{ij}^+ \propto q^{\frac{1}{2}} \displaystyle\sum_{\frac{i+1}{2} \le k \le \frac{j}{2}} \ell_{i,2k-1}^+\ \bar{\ell}_{2k,j}^+ - q^{-\frac{1}{2}} \sum_{\frac{i}{2} \le k \le \frac{j+1}{2}} \ell_{i,2k}^+\ \bar{\ell}_{2k-1,j}^+$ for $i \le j+1$ and 0 otherwise.

We have used property (4.8) and the relations between ℓ and $\bar{\ell}$. Similarly, it is possible to compute explicitly the charges B_{ij}^- in terms of ℓ_{ij}^- and $\bar{\ell}_{ij}^-$.

The charges $B^{\pm}_{ij}$ form an algebra, called $\mathcal{T}_f$ (f for finite), with the following defining exchange relations emerging from (2.13) as $\lambda_i \to \pm\infty$, i.e. we get

$$
\begin{aligned}
R_{12}\ \mathbb{B}^+_1\ \bar{R}_{12}\ \mathbb{B}^+_2 &= \mathbb{B}^+_2\ \bar{R}_{12}\ \mathbb{B}^+_1\ R_{12} &(4.12)\\
R_{12}\ \mathbb{B}^-_1\ R^{t_1}_{12}\ \mathbb{B}^-_2 &= \mathbb{B}^-_2\ R^{t_1}_{12}\ \mathbb{B}^-_1\ R_{12} &(4.13)\\
R_{12}\ \mathbb{B}^+_1\ \bar{R}_{12}\ \mathbb{B}^-_2 &= \mathbb{B}^-_2\ \bar{R}_{12}\ \mathbb{B}^+_1\ R_{12} &(4.14)\\
R_{12}\ \mathbb{B}^+_1\ R^{t_1}_{12}\ \mathbb{B}^-_2 &= \mathbb{B}^-_2\ R^{t_1}_{12}\ \mathbb{B}^+_1\ R_{12} &(4.15)
\end{aligned}
$$

where $\bar{R}_{12} = (R^{-1}_{21})^{t_1}$. In [26, 30], analogous results have been obtained in the SP case.

Remarks (1) The subalgebra, denoted $\mathcal{T}^+_f$, generated by B^+_{ij} is in fact isomorphic to the one, denoted $\mathcal{T}^-_f$, generated by B^-_{ij}: it is proved using the invertible transformation $\check{\mathbb{B}}^+_1 = M_1\left((\mathbb{B}^+_1)^{-1}\right)^t$ and showing that $\check{\mathbb{B}}^+$ satisfies relation (4.13).

(2) When $\mathcal{K}(\lambda)$ is the solution (i) (resp. solution (iv)), the subalgebra $\mathcal{T}^-_f$ is the twisted quantum algebra $U'_q(so_n)$ (resp. $U'_q(sp_n)$): these algebras were introduced in [31, 32].

(3) For the solution (iii), there are only two non-vanishing generators B^+_{1n} and B^-_{n1}. In this case, relations (4.12)-(4.15) reduce only to $[B^+_{1n}, B^-_{n1}] = 0$ which is an abelian algebra and is non-deformed.

(4) The algebra associated to the solution (ii) was discussed in [18], but here is treated in detail and gives interesting non trivial results as we shall see in the subsequent sections.

In the following, we will not study the case (iii) since it provides a trivial finite algebra.

4.2 Symmetry of the transfer matrix

We shall present in what follows the exchange relations among the finite algebras studied previously and the entries of the transfer matrix (i.e. $\mathrm{tr}_0\, \mathbb{B}_0(\lambda)$). Such relations will essentially manifest the existing –if any– symmetries of the model. More precisely, let us consider (2.13) for $\lambda_1 \to \pm\infty$. Bearing in mind that $R_{12}(\lambda \to +\infty) \propto R_{12}$ and $\bar{R}(\lambda \to +\infty) \propto \bar{R}_{12} = (R^{-1}_{21})^{t_1}$, one obtains the following set of exchange relations:

$$
R_{12}\mathbb{B}^+_1\bar{R}_{12}\mathbb{B}_2(\lambda) = \mathbb{B}_2(\lambda)\bar{R}_{12}\mathbb{B}^+_1 R_{12}. \tag{4.16}
$$

Similar expressions may be derived for the case that $\lambda \to -\infty$, but they are omitted here for brevity. Explicit expressions for the $\mathcal{U}_q(gl_3)$ case are provided in Appendix A.

Let us focus on the charges next to diagonal, their explicit expressions for solution (ii) coincide with the ones found in [18] i.e.

$$
B^+_{i,i+1} = \ell^+_{ii}\ \bar{\ell}^+_{i,i+1} + \ell^+_{i,i+1}\bar{\ell}^+_{i+1,i+1} + (q+1)\ell^+_{ii}\ \bar{\ell}^+_{i+1,i+1}. \tag{4.17}
$$

For $\mathcal{K} = \mathbb{I}$, the expression is the same with the last term vanishing. Let $c = q - q^{-1}$. Then,

from (4.16), we obtain:

$$
\begin{aligned}
& c\sum_l E_{i+1,l}\otimes B_{il}(\lambda) - qc\sum_l E_{i,l}\otimes B_{i+1,l}(\lambda) + \sum_{l,j} f_j E_{jl}\otimes B^+_{i,i+1}\ B_{jl}(\lambda) \\
&= c\sum_l E_{li}\otimes B_{l,i+1}(\lambda) - qc\sum_l E_{l,i+1}\otimes B_{l,i}(\lambda) + \sum_{l,j} \bar{f}_j E_{lj}\otimes B_{lj}(\lambda)\ B^+_{i,i+1}\,.
\end{aligned} \tag{4.18}
$$

From the latter equations, one concludes that $B^+_{i,i+1}$ do not commute with the SNP transfer matrix for any choice of left boundary (see also Appendix A for the full set of exchange relations for the $\mathcal{U}_q(gl_3)$ case) contrary to the SP case and to the rational SNP case where the exchange relations of the (4.16) allow the study of symmetry of the transfer matrix with boundary conditions. The higher order expansion of the transfer matrix provides naturally non-local quantities, which due to integrability commute with the transfer matrix, however such quantities being in involution form an abelian algebra. The main challenge within this context is the search of a non abelian algebra, which at the same time would form a symmetry of the transfer matrix as in the case of SP boundary conditions.

4.3 Quantum Brauer algebra

In this subsection, we basically recall the results obtained in [11]. The quantum deformation of the Brauer algebra $\mathcal{B}_N(z,q)$ [11] is generated by $\sigma_1,\ldots,\sigma_N,\tau_1$ subject to the following defining relations

$$
\begin{aligned}
\sigma_i^2 &= (q-q^{-1})\sigma_i + 1, \qquad \sigma_i\ \sigma_j = \sigma_j\ \sigma_i, \qquad \sigma_i\ \sigma_{i+1}\ \sigma_i = \sigma_{i+1}\ \sigma_i\ \sigma_{i+1}, \\
\tau_1^2 &= \frac{z-z^{-1}}{q-q^{-1}}\tau_1, \qquad \sigma_1\ \tau_1 = \tau_1\ \sigma_1 = q\tau_1 \\
\tau_1\ \sigma_2\ \tau_1 &= z\ \tau_1, \qquad \sigma_i\ \tau_1 = \tau_1\ \sigma_i, \qquad i=2,\ldots N-1 \\
\tau_1\ (zq\ \zeta^{-1} &+ z^{-1}q^{-1}\zeta)\ \tau_1\ (q\zeta^{-1}+q^{-1}\zeta) = (q\ \zeta^{-1}+q^{-1}\zeta)\ \tau_1\ (z\ q\ \zeta^{-1}+z^{-1}q^{-1}\zeta)\ \tau_1
\end{aligned} \tag{4.19}
$$

where $\zeta = \sigma_2\ \sigma_3\ \sigma_1\ \sigma_2$. Notice that the generators $\{\sigma_i\}$ form the usual Hecke algebra $\mathcal{H}_N(q)$. It has been shown in [11] that the map

$$
\sigma_i \mapsto U_{i,i+1} + q = \mathcal{P}_{i,i+1}R_{i,i+1} = \hat{R}_{i,i+1} \qquad \text{and} \quad \tau_1 \mapsto \mathcal{P}_{12}^{t_1}\,M_1^{-1} \tag{4.20}
$$

is a representation of $\mathcal{B}_N(q^n,q)$. The difference between the algebra defined here and the one introduced in [11] is such that, although we chose a different definition for the coproduct $\mathbb{T}(\lambda)$, the quantum Brauer algebra is still the centralizer of the quantum twisted Yangian.

As in the rational case, we focus on the fundamental representation of $U_q(\widehat{gl}_n)$ acting on $(\mathbb{C}^n)^{\otimes N}$ i.e.

$$
\mathbb{T}(\lambda) \mapsto R_{0N}(\lambda)\ldots R_{01}(\lambda)\,, \tag{4.21}
$$

which provides a representation of $\mathbb{B}(\lambda) \mapsto R_{0N}(\lambda)\dots R_{01}(\lambda)\,\mathcal{K}_0(\lambda)\,\bar{R}_{01}(\lambda)\dots\bar{R}_{0N}(\lambda)$. In particular, we get

$$\mathbb{B}_0^+ \quad\mapsto\quad R_{0N}\dots R_{01}\,\mathcal{K}_0^+\,\bar{R}_{01}\dots\bar{R}_{0N} = B_0^+ \tag{4.22}$$

$$\mathbb{B}_0^- \quad\mapsto\quad R_{N0}^{-1}\dots R_{10}^{-1}\,\mathcal{K}_0^-\,R_{01}^{t_0}\dots R_{0N}^{t_0} = B_0^- \tag{4.23}$$

where $\mathcal{K}^\pm = \mathcal{K}(\lambda\to\pm\infty)$ and we recall that $\bar{R}_{0i} = (R_{i0}^{-1})^{t_0}$.

A.I. Molev, in [11], focused on the solution (i) ($\mathcal{K}^\pm = \mathbb{I}$) i.e. $U'_q(so_n)$ and proved, in this case, that the actions of the quantum Brauer algebra $\mathcal{B}_N(q^n,q)$ (see (4.20)) on the space $(\mathbb{C}^n)^{\otimes N}$ and the one of the algebra $\mathcal{T}_f^-$, generated by B_{ij}^-, (see (4.23)) commute with each other i.e.

$$[U_{i,i+1}+q\,,\,B_0^-]=0 \qquad\text{and}\quad [\mathcal{P}_{12}^{t_1}\,M_1^{-1}\,,\,B_0^-]=0\,. \tag{4.24}$$

We can prove similarly the same result for the whole algebra $\mathcal{T}_f$ (generated by $B_{ij}^\pm$).

Unfortunately, it seems impossible to prove similar results for all the solutions $\mathcal{K}(\lambda)$. In addition, the matrices $\bar{R}$ and $\mathcal{K}^\pm$ cannot be written just using matrices similar to the ones representing σ_i and τ_i whereas it has been possible in the rational case (see end of section 3.3). These remarks indicate that we need a more general framework we will introduce in the following section.

4.4 Generalized quantum Brauer algebra and Baxterisation

It is well known that the trigonometric R matrix can be obtained from generators of the Hecke algebra: such an operation is called Baxterisation. Indeed, starting from $\hat{R}_{i,i+1}$ satisfying Hecke algebra, the following linear combination

$$\hat{R}_{i,i+1}(\lambda) = e^\lambda \hat{R}_{i,i+1} - e^{-\lambda}\hat{R}_{i,i+1}^{-1} \tag{4.25}$$

satisfies the braided Yang-Baxter equation

$$\hat{R}_{12}(\lambda_1-\lambda_2)\hat{R}_{23}(\lambda_1)\hat{R}_{12}(\lambda_2) = \hat{R}_{23}(\lambda_2)\hat{R}_{12}(\lambda_2)\hat{R}_{23}(\lambda_1-\lambda_2) \tag{4.26}$$

and unitarity condition $\hat{R}(\lambda)\hat{R}(-\lambda)\propto\mathbb{I}$. As explained before, it does not seem possible to construct $\bar{R}(\lambda)$ and $\mathcal{K}(\lambda)$ in a similar fashion. However, there exist equivalent forms where the Baxterisation for these two matrices becomes possible.

Let us introduce the conjugate index $\bar{k} = n+1-k$, the matrix $V = q^{k-\frac{N+1}{2}}E_{k\bar{k}}$ for n odd and $V = i(-1)^k q^{k-\frac{N+1}{2}}E_{k\bar{k}}$ for n even (such that $V^2=\mathbb{I}$ and $M=V^tV$) and $\widetilde{\mathbb{B}}_0(\lambda) = \mathbb{B}_0(\lambda)V_0^t$. Then, we get from equation (2.13), the following braided reflection equation

$$\hat{R}_{12}(\lambda_1-\lambda_2)\widetilde{\mathbb{B}}_1(\lambda_1)\widetilde{R}_{12}(\lambda_1+\lambda_2)\widetilde{\mathbb{B}}_1(\lambda_2) = \widetilde{\mathbb{B}}_1(\lambda_2)\widetilde{R}_{12}(\lambda_1+\lambda_2)\widetilde{\mathbb{B}}_1(\lambda_1)\hat{R}_{12}(\lambda_1-\lambda_2) \tag{4.27}$$

where

$$\widetilde{R}_{12}(\lambda) = \mathcal{P}_{12}V_2^t\bar{R}_{12}(\lambda)V_2^t = \mathcal{P}_{12}V_2^tR_{12}^{t_1}(-\lambda-i\rho)V_2^t \tag{4.28}$$

Obviously, the algebras generated by $\widetilde{\mathbb{B}}_0(\lambda)$ and $\mathbb{B}_0(\lambda)$ are isomorphic. Whereas for $\bar{R}(\lambda)$ there is no symmetric form, the matrix $\widetilde{R}(\lambda)$ can be written as

$$\widetilde{R}_{12}(\lambda) = e^{-\lambda-i\rho}\widetilde{R}_{12} - e^{\lambda+i\rho}\widetilde{R}_{12}^{-1} \tag{4.29}$$

where $\widetilde{R}_{12} = \mathcal{P}_{12}V_2^t R_{12}^{t_1} V_2^t$.

All numerical solutions $\widetilde{\mathcal{K}}(\lambda)$ of equation (4.27) can be obtained from the classification $\mathcal{K}(\lambda)$ using the relation $\widetilde{\mathcal{K}}(\lambda) = \mathcal{K}(\lambda)V^t$. The only solution which gives a non trivial finite algebra (see section 4.1) and depends on the spectral parameter is the solution (ii). The solution $\widetilde{\mathcal{K}}(\lambda)$ corresponding to the cases (ii) can be also written symmetrically as

$$\widetilde{\mathcal{K}}(\lambda) = e^{\lambda}q^{\frac{n}{4}}\widetilde{\mathcal{K}} \pm e^{-\lambda}q^{-\frac{n}{4}}\widetilde{\mathcal{K}}^{-1} \tag{4.30}$$

where $\widetilde{\mathcal{K}} = \left(q^{\frac{1}{2}} + q^{-\frac{1}{2}}\right)\left(\frac{q^{-1}}{q^{-1}+1}\sum_i \epsilon_i^2 E_{ii} + \sum_{i<j}\epsilon_i\epsilon_j E_{ij}\right)V^t$.

We are now interested in the reciprocity: we want to find an algebra whose representation gives $\hat{R}$, $\widetilde{R}$ and $\widetilde{\mathcal{K}}$ and which is sufficient to prove the different relations (Yang-Baxter equation, unitarity, reflection equation) that the matrices $\hat{R}(\lambda)$, $\widetilde{R}(\lambda)$ and $\widetilde{\mathcal{K}}(\lambda)$ of the forms (4.25), (4.29) and (4.30) must satisfy. Let us define a new algebra, called $\mathcal{N}_N(w,\pm,q)$, generated by $\{\sigma_i\}$ satisfying the relation of the Hecke algebra and by the invertible generators $\{\rho_i\}$ and $\{b_i\}$ subject to the following defining relations,

$$\rho_i(\rho_i - \rho_i^{-1}) = \pm w^2(\rho_i - \rho_i^{-1}) \tag{4.31}$$
$$\sigma_i b_j = b_j\sigma_i \quad , \quad \rho_i b_j = b_j\rho_i \qquad \text{for } i>j \text{ or } i<j-1 \tag{4.32}$$
$$\sigma_i\rho_j = \rho_j\sigma_i \quad , \quad \rho_i\rho_j = \rho_j\rho_i \qquad \text{for } |i-j|>1 \tag{4.33}$$
$$\sigma_{i+1}\rho_i\rho_{i+1} = \rho_i\rho_{i+1}\sigma_i \quad , \quad \sigma_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\sigma_{i+1} \tag{4.34}$$
$$\sigma_{i+1}\rho_i^{-1}\rho_{i+1} = \rho_i\rho_{i+1}^{-1}\sigma_i \quad , \quad \sigma_i\rho_{i+1}^{-1}\rho_i = \rho_{i+1}\rho_i^{-1}\sigma_{i+1} \tag{4.35}$$
$$\sigma_i\rho_{i+1}\rho_i^{-1} - \sigma_i^{-1}\rho_{i+1}^{-1}\rho_i = \rho_{i+1}^{-1}\rho_i\sigma_{i+1} - \rho_{i+1}\rho_i^{-1}\sigma_{i+1}^{-1} \tag{4.36}$$
$$\sigma_{i+1}\rho_i\rho_{i+1}^{-1} - \sigma_{i+1}^{-1}\rho_i^{-1}\rho_{i+1} = \rho_i^{-1}\rho_{i+1}\sigma_i - \rho_i\rho_{i+1}^{-1}\sigma_i^{-1} \tag{4.37}$$
$$\sigma_i b_i\rho_i^{-1}b_i = b_i\rho_i^{-1}b_i\sigma_i \quad , \quad \rho_i b_i\sigma_i b_i = b_i\sigma_i b_i\rho_i \tag{4.38}$$
$$\sigma_i b_i\rho_i b_i - \sigma_i b_i^{-1}\rho_i^{-1}b_i^{-1} = b_i\rho_i b_i\sigma_i - b_i^{-1}\rho_i^{-1}b_i^{-1}\sigma_i \tag{4.39}$$
$$\sigma_i b_i^{-1}\rho_i b_i - \sigma_i^{-1}b_i\rho_i b_i^{-1} = b_i\rho_i b_i^{-1}\sigma_i - b_i^{-1}\rho_i b_i\sigma_i^{-1} \tag{4.40}$$
$$\sigma_i b_i^{-1}\rho_i^{-1}b_i - \sigma_i^{-1}b_i\rho_i^{-1}b_i^{-1} = b_i\rho_i^{-1}b_i^{-1}\sigma_i - b_i^{-1}\rho_i^{-1}b_i\sigma_i^{-1} \tag{4.41}$$

We define also two quotients of $\mathcal{N}_N(w,\pm,q)$, called $\mathcal{N}_N^0(w,\pm,q)$ (resp. $\mathcal{N}_N^1(w,\pm,q)$), defined by the supplementary relations

$$b_i^2 = 1 \quad (\text{resp. } (b_i + \sqrt{\pm 1}w^{-1})(b_i + b_i^{-1}) = 0\ ), \tag{4.42}$$

where we recall that we used the convention $\sqrt{1} = 1$ and $\sqrt{-1} = i$. Finally, let us introduce

$$\hat{r}_{i,i+1}(\lambda) = e^{\lambda}\sigma_i - e^{-\lambda}\sigma_i^{-1} \quad , \quad \widetilde{r}_{i,i+1}(\lambda) = w^{-1}e^{-\lambda}\rho_i - we^{\lambda}\rho_i^{-1} \tag{4.43}$$

and

$$k_i(\lambda) = \begin{cases} b_i \ , \ \text{for } \mathcal{N}_N^0(w,\pm,q) \\ e^{\lambda} w^{\frac{1}{2}} b_i + \eta e^{-\lambda} w^{-\frac{1}{2}} b_i^{-1} \ , \ \text{for } \mathcal{N}_N^1(w,\pm,q) \, , \end{cases} \tag{4.44}$$

where $\eta = \pm$ is a supplementary freedom. These algebras allow us to obtain the following proposition solving essentially the Baxterisation problem in the case of the quantum twisted Yangian:

Proposition 2 *The generators $\hat{r}_{i,i+1}(\lambda)$ satisfy the braided Yang-Baxter equation. We get also*

$$\hat{r}_{12}(\lambda_1-\lambda_2)\widetilde{r}_{23}(\lambda_1)\widetilde{r}_{12}(\lambda_2) = \widetilde{r}_{23}(\lambda_2)\widetilde{r}_{12}(\lambda_2)\hat{r}_{23}(\lambda_1-\lambda_2)\,. \tag{4.45}$$

The generators $\hat{r}_{i,i+1}(\lambda)$, $\widetilde{r}_{i,i+1}(\lambda)$ and $k_i(\lambda)$ satisfy the braided reflection equation

$$\hat{r}_{12}(\lambda_1-\lambda_2)k_1(\lambda_1)\widetilde{r}_{12}(\lambda_1+\lambda_2)k_1(\lambda_2) = k_1(\lambda_2)\widetilde{r}_{12}(\lambda_1+\lambda_2)k_1(\lambda_1)\hat{r}_{12}(\lambda_1-\lambda_2)\,. \tag{4.46}$$

We get also the following unitarity conditions

$$\frac{\hat{r}_{12}(\lambda)\hat{r}_{12}(-\lambda)}{(qe^{\lambda}-q^{-1}e^{-\lambda})(qe^{-\lambda}-q^{-1}e^{\lambda})} = 1 \quad , \quad \frac{\widetilde{r}_{12}(\lambda)\widetilde{r}_{12}(-\lambda)}{(we^{\lambda}-w^{-1}e^{-\lambda})(we^{-\lambda}-w^{-1}e^{\lambda})} = 1\,. \tag{4.47}$$

The unitarity condition for $k(\lambda)$ depends on the choice of the quotient: for $\mathcal{N}_N^0(w,\pm,q)$, we get $k_1(\lambda)k_1(-\lambda)=1$, for $\mathcal{N}_N^1(w,-,q)$,

$$k_1(\lambda)k_1(-\lambda) = \eta e^{2\lambda} + \eta e^{-2\lambda} - w - w^{-1} \ , \tag{4.48}$$

and, finally, for $\mathcal{N}_N^1(w,+,q)$,

$$k_1(\lambda)k_1(-\lambda+i\frac{\pi}{2}) = i\left(\eta e^{-2\lambda} - \eta e^{2\lambda} - w + w^{-1}\right)\,. \tag{4.49}$$

Proof. This proposition is proved by direct computation expressing the generators $\hat{r}_{i,i+1}(\lambda)$, $\widetilde{r}_{i,i+1}(\lambda)$ and $k_i(\lambda)$ thanks to (4.43)-(4.44) and using the defining exchange relations (4.31)-(4.42) of the algebras $\mathcal{N}_N(w,\pm,q)$, $\mathcal{N}_N^0(w,\pm,q)$ or $\mathcal{N}_N^1(w,\pm,q)$. For example, we get

$$\begin{aligned} \widetilde{r}_{12}(\lambda)\widetilde{r}_{12}(-\lambda) &= \left(w^{-1}e^{-\lambda}\rho_1 - we^{\lambda}\rho_1^{-1}\right)\left(w^{-1}e^{\lambda}\rho_1 - we^{-\lambda}\rho_1^{-1}\right) \\ &= w^{-2}\rho_1^2 + w^2\rho_1^{-2} - e^{-2\lambda} - e^{2\lambda} \\ &= w^{-2}\rho_1(\rho_1-\rho_1^{-1}) + w^{-2} - w^2\rho_1^{-1}(\rho_1-\rho_1^{-1}) + w^2 - e^{-2\lambda} - e^{2\lambda} \\ &= w^{-2} + w^2 - e^{-2\lambda} - e^{2\lambda}\,, \qquad \text{using relation (4.31)} \quad , \end{aligned}$$

which proves the second relation of (4.47). ∎

The link between the generalized quantum Brauer algebra and the numerical previous solutions is given by the representation of $\mathcal{N}_N^0(w,\pm,q)$ or $\mathcal{N}_N^1(w,\pm,q)$. The map

$$\sigma_i \mapsto \hat{R}_{i,i+1} \quad , \quad \rho_i \mapsto \widetilde{R}_{i,i+1} \quad \text{and} \quad b_i \mapsto V_i^t \text{ or } i(\mathcal{G}_q^-)_i\, V_i^t \tag{4.50}$$

is a representation of $\mathcal{N}_N^0(q^{\frac{n}{2}},(-1)^{n+1},q)$ (we recall that $q^{\frac{n}{2}}=e^{i\rho}$). It is proved by direct calculation. We then obtain the numerical matrices associated to the solutions (i) or (iv). Similarly, we can prove that

$$\sigma_i\mapsto\hat{R}_{i,i+1}\qquad,\quad\rho_i\mapsto\widetilde{R}_{i,i+1}\qquad\text{and}\quad b_i\mapsto\widetilde{\mathcal{K}}_i\ \text{(given after (4.30))}\tag{4.51}$$

is a representation of $\mathcal{N}_N^1(q^{\frac{n}{2}},(-1)^{n+1},q)$. In this case, we obtain the numerical solutions associated to solution (ii). Recall that we do not study the case (iii) since it provides a trivial finite algebra.

4.5 Quantum Brauer duality

We establish a result similar to the one of subsection 4.3 but now for any generic solution $\mathcal{K}(\lambda)$ using the framework of the previous subsection.

We can define a finite algebra $\widetilde{\mathcal{T}}_f$ generated by $\widetilde{\mathbb{B}}_0^{\pm}=\mathbb{B}_0^{\pm}V_0^t$, which is obviously isomorphic to $\mathcal{T}_f$. These generators can be represented by

$$\widetilde{\mathbb{B}}_0^{+}\mapsto\widetilde{B}_0^{+}=B_0^{+}V_0^t=\hat{R}_{N0}\hat{R}_{N-1,N}\dots\hat{R}_{23}\hat{R}_{12}\ \widetilde{K}_1\ \widetilde{R}_{12}^{-1}\widetilde{R}_{23}^{-1}\dots\widetilde{R}_{N-1,N}^{-1}\widetilde{R}_{N0}^{-1}\tag{4.52}$$

$$\widetilde{\mathbb{B}}_0^{-}\mapsto\widetilde{B}_0^{-}=B_0^{-}V_0^t=\hat{R}_{N0}^{-1}\hat{R}_{N-1,N}^{-1}\dots\hat{R}_{23}^{-1}\hat{R}_{12}^{-1}\ \widetilde{K}_1^{-1}\ \widetilde{R}_{12}\widetilde{R}_{23}\dots\widetilde{R}_{N-1,N}\widetilde{R}_{N0}\tag{4.53}$$

where $\widetilde{K}=V^t$ for (i), $\widetilde{K}=i\mathcal{G}_q^-V^t$ for (iv) and $\widetilde{K}$ is given just after (4.30) for (ii). Recall that we do not investigate the case (iii) which gives a trivial finite algebra. Let us remark that with this form of the algebra, we succeed in writing the generators $\widetilde{B}^{\pm}$ in terms of matrices appearing also in the representation (see (4.50) or (4.51)) of the algebra $\mathcal{N}_N(w,\pm,q)$.

The following proposition gives the analogue of the Brauer duality between the algebra generated by the elements of $\widetilde{B}_0^{\pm}$ and subalgebra of $\mathcal{N}_N(q^{\frac{n}{2}},\pm,q)$:

Proposition 3 *Let us introduce* $g_1=b_1^{-1}\,\dfrac{\rho_1-\rho_1^{-1}}{q-q^{-1}}\,b_1$*. The actions of the algebra generated by* $\{\sigma_i|1\leq i\leq N-1\}$ *and* g_1 *(see (4.50)-(4.51)) and of the algebra* $\widetilde{\mathcal{T}}_f$ *on the space* $(\mathbb{C}^n)^{\otimes N}$ *commute with each other.*

Proof. The proof of this proposition is based in showing the following two relations, for $1\leq i\leq N-1$,

$$\left[\hat{R}_{i,i+1}\,,\,\widetilde{B}_0^{\pm}\right]=0\qquad\text{and}\qquad\left[\widetilde{K}_1^{-1}\left(\widetilde{R}_{12}-\widetilde{R}_{12}^{-1}\right)\widetilde{K}_1\,,\,\widetilde{B}_0^{\pm}\right]=0\tag{4.54}$$

We shall use the fact that all the matrices involved in this computation satisfy the exchange relations of $\mathcal{N}_N(w,\pm,q)$. Let us give explicitly the proof for $[\widetilde{K}_1^{-1}\left(\widetilde{R}_{12}-\widetilde{R}_{12}^{-1}\right)\widetilde{K}_1,\widetilde{B}_0^{+}]=0$. Since $\widetilde{K}_1$ and $\widetilde{R}_{12}$ commute with matrices in space $0,3,4,\dots,N$, we can focus on proving

$$\left[\widetilde{K}_1^{-1}\left(\widetilde{R}_{12}-\widetilde{R}_{12}^{-1}\right)\widetilde{K}_1,\hat{R}_{23}\hat{R}_{12}\ \widetilde{K}_1\ \widetilde{R}_{12}^{-1}\widetilde{R}_{23}^{-1}\right]=0\,.\tag{4.55}$$

Let us first compute

$$\begin{aligned}\widetilde{K}_1^{-1}\widetilde{R}_{12}\ (\widetilde{K}_1\hat{R}_{23})\ \hat{R}_{12}\widetilde{K}_1\widetilde{R}_{12}^{-1}\widetilde{R}_{23}^{-1} &= \widetilde{K}_1^{-1}\widetilde{R}_{12}\hat{R}_{23}\ (\widetilde{K}_1\hat{R}_{12}\widetilde{K}_1\widetilde{R}_{12}^{-1})\ \widetilde{R}_{23}^{-1}\\ &= \widetilde{K}_1^{-1}\ (\widetilde{R}_{12}\hat{R}_{23}\widetilde{R}_{12}^{-1})\ \widetilde{K}_1\hat{R}_{12}\widetilde{R}_{23}^{-1}\widetilde{K}_1\\ &= \widetilde{R}_{23}^{-1}\widetilde{K}_1^{-1}\hat{R}_{12}\widetilde{K}_1\ (\widetilde{R}_{23}\hat{R}_{12}\widetilde{R}_{23}^{-1})\ \widetilde{K}_1\\ &= \widetilde{R}_{23}^{-1}\ (\widetilde{K}_1^{-1}\hat{R}_{12}\widetilde{K}_1\widetilde{R}_{12}^{-1})\ \hat{R}_{23}\widetilde{R}_{12}\widetilde{K}_1\\ &= \widetilde{R}_{23}^{-1}\widetilde{R}_{12}^{-1}\widetilde{K}_1\hat{R}_{12}\hat{R}_{23}\quad \widetilde{K}_1^{-1}\widetilde{R}_{12}\widetilde{K}_1 \qquad (4.56)\end{aligned}$$

In the latter relations, the parenthesis indicate which part of the product is modified using different relations (4.31)-(4.41). We obtained similar result for $\widetilde{K}_1^{-1}\widetilde{R}_{12}^{-1}\widetilde{K}_1$. Then, we have proved that

$$\widetilde{K}_1^{-1}\left(\widetilde{R}_{12}-\widetilde{R}_{12}^{-1}\right)\widetilde{K}_1\widetilde{B}_0^{+} = \widetilde{C}_0^{+}\widetilde{K}_1^{-1}\left(\widetilde{R}_{12}-\widetilde{R}_{12}^{-1}\right)\widetilde{K}_1, \qquad (4.57)$$

where $\widetilde{C}_0^{+} = \hat{R}_{N0}\hat{R}_{N-1,N}\ldots\hat{R}_{34}\widetilde{R}_{23}^{-1}\ \widetilde{R}_{12}^{-1}\widetilde{K}_1\ \hat{R}_{12}\hat{R}_{23}\widetilde{R}_{34}^{-1}\ldots\widetilde{R}_{N-1,N}^{-1}\widetilde{R}_{N0}^{-1}$. Multiplying on the left by $\widetilde{K}_1^{-1}\widetilde{R}_{12}\widetilde{K}_1$ the previous relation and knowing that it satisfies relation (4.31), we finish the proof. ∎

Within this framework, the proof of this proposition requires only the exchange relations of the algebra $\mathcal{N}(w,\pm,q)$ whereas, in the previous case (see subsection 4.3), the proof of the analogous proposition is based on direct computation.

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A Appendix

In this appendix, we give the explicit expression of the commutation relations between the finite algebra and the quantum twisted Yangian for $\mathcal{U}_q(gl_3)$ and for $\mathcal{K}(\lambda)=\mathbb{I}$. Let us express $\mathbb{B}$ as a matrix

$$\mathbb{B}(\lambda) = \begin{pmatrix} \mathcal{A}_1 & \mathcal{D}_1 & \mathcal{D} \\ \mathcal{C}_1 & \mathcal{A}_2 & \mathcal{D}_2 \\ \mathcal{C} & \mathcal{C}_2 & \mathcal{A}_3 \end{pmatrix}. \qquad (A.1)$$

Set also $c = q - q^{-1}$ and $[X,\ Y]_q = qXY - q^{-1}YX$. We recall that the diagonal charges are trivial $B_{ii}^{+} \propto \mathbb{I}$ so we simply deal with the rest of them. Then from (4.16) and bearing in mind (A.1) various sets of equations are inferred:

$$\begin{aligned}\Big[B_{12}^{+},\mathcal{A}_1\Big]_q &= qc\mathcal{C}_1 + c\mathcal{D}_1, \qquad \Big[B_{12}^{+},\mathcal{A}_2\Big]_{q^{-1}} = -qc\mathcal{C}_1 - c\,\mathcal{D}_1, \qquad \Big[B_{12}^{+},\mathcal{A}_3\Big] = 0,\\ q\Big[B_{12}^{+},\ \mathcal{D}_1\Big] &= \Big[B_{12}^{+},\ \mathcal{C}_1\Big] = qc(\mathcal{A}_2 - \mathcal{A}_1), \qquad \Big[B_{12}^{+},\ \mathcal{D}_2\Big]_{q^{-\frac{1}{2}}} = -q^{\frac{1}{2}}c\mathcal{D}\\ \Big[B_{12}^{+},\ \mathcal{C}_2\Big]_{q^{-\frac{1}{2}}} &= -q^{\frac{1}{2}}c\mathcal{C}_1, \qquad \Big[B_{11}^{+},\ \mathcal{D}\Big]_{q^{\frac{1}{2}}} = q^{\frac{1}{2}}c\mathcal{D}_2, \qquad \Big[B_{12}^{+},\ \mathcal{C}\Big]_{q^{\frac{1}{2}}} = q^{\frac{1}{2}}c\mathcal{C}_2 \qquad (A.2)\end{aligned}$$

$$
\begin{aligned}
\Big[B_{23}^{+},\ \mathcal{A}_1\Big] &= 0, \quad \Big[B_{23}^{+},\ \mathcal{A}_2\Big]_q = q\ c\ \mathcal{C}_2 + c\mathcal{D}_2, \quad [B_{23}^{+},\ \mathcal{A}_3\Big]_{q^{-1}} = -qc\mathcal{C}_2 - c\ \mathcal{D}_2 \\
q\Big[B_{22}^{+},\ \mathcal{D}_2\Big] &= \Big[B_{23}^{+},\ \mathcal{C}_2\Big] = qc(\mathcal{A}_3 - \mathcal{A}_2), \quad \Big[B_{23}^{+},\ \mathcal{D}_1\Big]_{q^{\frac{1}{2}}} = q^{\frac{1}{2}} c\mathcal{D} \\
\Big[B_{23}^{+},\ \mathcal{C}_1\Big]_{q^{\frac{1}{2}}} &= q^{\frac{1}{2}} c\mathcal{C}, \quad \Big[B_{23}^{+},\ \mathcal{D}\Big]_{q^{-\frac{1}{2}}} = -q^{\frac{1}{2}} c\mathcal{D}_1, \quad \Big[B_{23}^{+},\ \mathcal{C}\Big]_{q^{-\frac{1}{2}}} = -q^{\frac{1}{2}} c\mathcal{C}_1 \qquad \text{(A.3)}
\end{aligned}
$$

$$
\begin{aligned}
\Big[B_{13}^{+},\ \mathcal{A}_1\Big]_q &= qc\mathcal{C} + c\mathcal{D}, \quad \Big[B_{13}^{+},\ \mathcal{A}_3\Big]_{q^{-1}} = -qc\mathcal{C} - c\ \mathcal{D}, \\
q\Big[B_{13}^{+},\ \mathcal{D}\Big] &= \Big[B_{13}^{+},\ \mathcal{C}\Big] = qc(\mathcal{A}_3 - \mathcal{A}_1), \quad \Big[B_{13}^{+},\ \mathcal{D}_1\Big]_{q^{\frac{1}{2}}} = c(q^{\frac{1}{2}}\mathcal{C}_2 + q^{-\frac{1}{2}}\mathcal{D}B_{12}^{+} - q^{-\frac{1}{2}}\mathcal{A}_1 B_{23}^{+}) \\
\Big[B_{13}^{+},\ \mathcal{C}_1\Big]_{q^{\frac{1}{2}}} &= q^{\frac{1}{2}} c(\mathcal{D}_2 + B_{12}^{+}\mathcal{C} - B_{23}^{+}\mathcal{A}_1), \quad \Big[B_{13}^{+},\ \mathcal{D}_2\Big]_{q^{-\frac{1}{2}}} = c(-q^{\frac{1}{2}}\mathcal{C}_1 + q^{-\frac{1}{2}} B_{12}^{+}\mathcal{A}_3 - q^{-\frac{1}{2}} B_{23}^{+}\mathcal{D}), \\
\Big[B_{13}^{+},\ \mathcal{C}_2\Big]_{q^{-\frac{1}{2}}} &= q^{\frac{1}{2}} c(\mathcal{A}_3 B_{12} - \mathcal{C}B_{23} - \mathcal{D}_1), \quad \Big[B_{13}^{+},\ \mathcal{A}_2\Big] = c(-B_{23}^{+}\mathcal{D}_1 - \mathcal{C}_1 B_{23}^{+} + \mathcal{D}_2 B_{12}^{+} + B_{12}^{+}\mathcal{C}_2).
\end{aligned}
$$

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